MATH 3235 Probability Theory 2/6/22

$X : \Omega \to \mathbb{R}$ discrete r.v.

a) $Im\{X\}$ is finite or countable

b) $\{\omega \in \Omega \mid X(\omega) = x\} \in \mathcal{F}$

$x \in Im(X)$

From b) we can define

$$p(x) = \mathbb{P}(\{\omega \mid X(\omega) = x\})$$

$p : Im(X) \to \mathbb{R}^+$

p (p_X) probability mass function of X

p.m.f.

$Im(X)$ p_X Then X is completely given.

Thm: x_i $\quad i=1 \dots \infty$ $\quad x_i \in \mathbb{R}$

π_i $\quad i=1 \dots \infty$ $\quad \pi_i \geq 0$

$$\sum_{i=1}^{\infty} \pi_i = 1$$

There exists a r.v. X such that

$$Im(X) = \{x_i \mid i=1 \dots \infty\}$$

$$p(x_i) = \pi_i$$

Proof:

$$\Omega = \{x_i \mid i=1 \dots \infty\}$$

$\mathcal{F}$ = power set of Ω

$$\mathbb{P}(\{x_i\}) = \pi_i$$

$$X : \Omega \to \mathbb{R} \qquad X(x_i) = x_i$$

Examples.

Coin flip. We flip a coin N times

H T p of getting H

$(1-p)=q$ of getting T

$|\Omega| = 2^N$ $\mathbb{P}(\{\omega\}) = p^{h(\omega)} q^{N-h(\omega)}$

$h(\omega) = \#$ of H in Ω

$t(\omega) = N - h(\omega)$ $\#$ of T

$$X_i(\omega) = \begin{cases} 1 & \text{if the i-th flip } H \\ 0 & \text{" " " " } T \end{cases}$$

X_i Bernoulli r.v.

$$p_{X_i}(1) = p \sum_{\omega : \omega_i = 1} p^{h(\omega)-1} q^{N-h(\omega)} = p$$

$$p_{X_i}(0) = q$$

X_i is Bernoulli r.v. with par. p

$X_i \sim \mathrm{Ber}(p)$

$$\mathbb{P}(X_i = 1 \,\&\, X_j = 0) = p\,q \quad \text{if } i \neq j$$

$$S = \sum_{i=1}^{N} X_i$$

$S(\omega)$ number of H in ω

$$\mathbb{P}(S = s)$$

$$S(\omega) = s \implies \mathbb{P}(\{\omega\}) = p^s q^{N-s}$$

$$\#\{\omega \mid S(\omega) = s\} = \binom{N}{s} = \frac{N!}{s!\,(N-s)!}$$

$$0! = 1$$

$$P_S(s) = \binom{N}{s} p^s (1-p)^{N-s}$$

Binomial r.v with par N, p

$$S \sim \text{Bin}(N, s)$$

$$P_S(s) \geq 0$$

$$\sum_{s=0}^{N} p(s) = 1$$

$$\boxed{\sum (x+y)^N = \sum_{n=0}^{N} \binom{N}{n} x^n y^{N-n}}$$

$$\sum_{s=0}^{N} p(s) = (p+q)^N = 1$$

Queuing Theory.

Number of requests in 1h ?

N

m very large elementary intervals

$1/m$

a) In any interval I have at most 1 request.

b) Number of request in The intervalls are independent

c) Probability of one arrival in an intervall is λ/m

$$P_N(n) = \mathbb{P}(N=n) = \binom{m}{n}\left(\frac{\lambda}{m}\right)^n\left(1-\frac{\lambda}{m}\right)^{m-n}$$

$$\frac{m!}{(m-n)!} = m(m-1)\dots(m-n+1) \simeq m^n$$

$$\left(1-\frac{\lambda}{m}\right)^m \simeq e^{-\lambda}$$

$$\frac{\cancel{m^n}}{n!}\left(\frac{\lambda}{\cancel{m}}\right)^n e^{-\lambda\left(1-\frac{n}{m}\right)} \simeq \frac{\lambda^n}{n!}e^{-\lambda}$$

$$\lim_{m\to\infty}\binom{m}{n}\left(\frac{\lambda}{m}\right)^n\left(1-\frac{\lambda}{m}\right)^{m-n} = \frac{\lambda^n}{n!}e^{-\lambda}$$

N $$P_N(n) = \frac{\lambda^n}{n!}e^{-\lambda}$$

N is a Poisson r.v. with par λ

Number of arrivals in $\frac{1}{2}$ h.

$N_{\frac{1}{2}} \sim \mathrm{Poi}\left(\frac{\lambda}{2}\right)$

$N_t \sim Poi(\lambda t)$

$N_t - N_s$ = arrivals between s and t

$t \geq s$.

$N_t - N_s \sim Poi(\lambda(t-s))$

$[s_1, t_1]$ and $[s_2, t_2]$ with

$s_2 > t_1$

arrivals in $[s_1, t_1]$ $\perp\!\!\!\perp$

arrivals in $[s_2, t_2]$

Poisson Process.

Flip the coin until you get

the firs H

N Total number

$P_N(n) = q^{n-1} p$ Geometric r.v.

Hypergeometric.

10 people out of a population

How many male?

Population N male

M female

$N+M$ Total

I select n.

X number of male

$$P_X(x) = \frac{\binom{N}{x}\binom{M}{n-x}}{\binom{M+N}{n}}$$

$$M \to \infty \qquad \frac{M}{N+M} \to p$$